Quantum Hall effect and Hierarchy of Planck Constants

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December 5, 2008

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Abstract

I have already earlier proposed the explanation of FQHE, anyons, and fractionization of quantum numbers in terms of hierarchy of Planck constants realized as a generalization of the imbedding space $H = M^4 \times CP_2$ to a book like structure. The book like structure applies separately to CP_2 and to causal diamonds ($CD \subset M^4$) defined as intersections of future and past directed light-cones. The pages of the Big Book correspond to singular coverings and factor spaces of CD (CP_2) glued along 2-D subspace of CD (CP_2) and are labeled by the values of Planck constants assignable to CD and CP_2 and appearing in Lie algebra commutation relations. The observed Planck constant $\hbar$, whose square defines the scale of M^4 metric corresponds to the ratio of these Planck constants. The key observation is that fractional filling factor results if $\hbar$ is scaled up by a rational number.

In this chapter I try to formulate more precisely this idea. The outcome is a rather detailed view about anyons on one hand, and about the Kähler structure of the generalized imbedding space on the other hand.

1. Fundamental role is played by the assumption that the Kähler gauge potential of CP_2 contains a gauge part with no physical implications in the context of gauge theories but contributing to physics in TGD framework since $U(1)$ gauge transformations are representations of symplectic transformations of CP_2. Also in the case of CD it makes also sense to speak about Kähler gauge potential. The gauge part codes for Planck constants of CD and CP_2 and leads to the identification of anyons as states associated with partonic 2-surfaces surrounding the tip of CD and fractionization of quantum numbers. Explicit formulas relating fractionized charges to the coefficients characterizing the gauge parts of Kähler gauge potentials of CD and CP_2 are proposed based on some empirical input.

2. One important implication is that Poincare and Lorentz invariance are broken inside given CD although they remain exact symmetries at the level of the geometry of world of classical worlds (WCW). The interpretation is as a breaking of symmetries forced by the selection of quantization axis.

3. Anyons would basically correspond to matter at 2-dimensional "partonic" surfaces of macroscopic size surrounding the tip of the light-cone boundary of CD and could be regarded as gigantic elementary particle states with very large quantum numbers and by charge fractionization confined around the tip of CD. Charge fractionization

and anyons would be basic characteristic of dark matter (dark only in relative sense). Hence it is not surprising that anyons would have applications going far beyond condensed matter physics. Anyonic dark matter concentrated at 2-dimensional surfaces would play key key role in the the physics of stars and black holes, and also in the formation of planetary system via the condensation of the ordinary matter around dark matter. This assumption was the basic starting point leading to the discovery of the hierarchy of Planck constants. In living matter membrane like structures would represent a key example of anyonic systems as the model of DNA as topological quantum computer indeed assumes.

4. One of the basic questions has been whether TGD forces the hierarchy of Planck constants realized in terms of generalized imbedding space or not. The condition that the choice of quantization axes has a geometric correlate at the imbedding space level motivated by quantum classical correspondence of course forces the hierarchy: this has been clear from the beginning. It is now clear that first principle description of anyons requires the hierarchy in TGD Universe. The hierarchy reveals also new light to the huge vacuum degeneracy of TGD and reduces it dramatically at pages for which CD corresponds to a non-trivial covering or factor space, which suggests that mathematical existence of the theory necessitates the hierarchy of Planck constants. Also the proposed manifestation of Equivalence Principle at the level of symplectic fusion algebras as a duality between descriptions relying on the symplectic structures of CD and CP_2 forces the hierarchy of Planck constants.

Keywords: Topological Geometrodynamics, quantum biology, topological quantum computation, DNA.

1 Introduction

Quantum Hall effect [29, 30, 35] occurs in 2-dimensional systems, typically a slab carrying a longitudinal voltage V causing longitudinal current j. A magnetic field orthogonal to the slab generates a transversal current component j_T by Lorentz force. j_T is proportional to the voltage V along the slab and the dimensionless coefficient is known as transversal conductivity. Classically the coefficients is proportional ne/B, where n is 2-dimensional electron density and should have a continuous spectrum. The finding that came as surprise was that the change of the coefficient as a function of parameters like magnetic field strength and temperature occurred as discrete steps of same size. In integer quantum Hall effect the coefficient is quantized to $2\nu\alpha$, $\alpha = e^2/4\pi$, such that ν is integer.

Later came the finding that also smaller steps corresponding to the filling fraction $\nu = 1/3$ of the basic step were present and could be understood if the charge of electron would have been replaced with $\nu = 1/3$ of its ordinary value. Later also QH effect with wide large range of filling fractions of form $\nu = k/m$ was observed.

The model explaining the QH effect is based on pseudo particles known as anyons [39, 30]. According to the general argument of [28] anyons have fractional charge νe. Also the TGD based model for fractionization to be discussed later suggests that the anyon charge should be νe quite generally. The braid statistics of anyon is believed to be fractional so that anyons are neither bosons nor fermions. Non-fractional statistics is absolutely essential for the vacuum degeneracy used to represent logical qubits.

In the case of Abelian anyons the gauge potential corresponds to the vector potential of the divergence free velocity field or equivalently of incompressible anyon current. For non-Abelian anyons the field theory defined by Chern-Simons action is free field theory and in well-defined sense trivial although it defines knot invariants. For non-Abelian anyons situation would be different. They would carry non-Abelian gauge charges possibly related to a symmetry breaking to a discrete subgroup H of gauge group [39] each of them defining an incompressible hydrodynamical flow. According to [18] the anyons associated with the filling fraction $\nu = 5/2$ are a good candidate for non-Abelian anyons and in this case the charge of electron is reduced to $Q = e/4$ rather than being $Q = \nu e$ [36]. This finding favors non-Abelian models [35].

Non-Abelian anyons [38, 30] are always created in pairs since they carry a conserved topological charge. In the model of [18] this charge should have values in 4-element group Z_4 so that it is conserved only modulo 4 so that charges +2 and -2 are equivalent as are also charges 3 and -1. The state of n anyon pairs created from vacuum can be show to possess 2^{n-1}-dimensional vacuum degeneracy [37]. When two anyons fuse the 2^{n-1}-dimensional state space decomposes to 2^{n-2}-dimensional tensor factors corresponding to anyon Cooper pairs with topological charges 2 and 0. The topological "spin" is ideal for representing logical qubits. Since free topological charges are not possible the notion of physical qubit does not make sense (note the analogy with quarks). The measurement of topological qubit reduces to a measurement of whether anyon Cooper pair has vanishing topological charge or not.

Topological quantum computation is perhaps the most promising application of anyons [17, 18, 19, 20, 21, 22, 23].

I have already earlier proposed the explanation of FQHE, anyons, and fractionization of quantum numbers in terms of hierarchy of Planck constants realized as a generalization of the imbedding space $H = M^4 \times CP_2$ to a book like structure [A9]. The book like structure applies separately to CP_2 and to causal diamonds ($CD \subset M^4$) defined as intersections of future and past directed light-cones. The pages of the Big Book correspond to singular coverings and factor spaces of CD (CP_2) glued along 2-D subspace of CD (CP_2) and are labeled by the values of Planck constants assignable to CD and CP_2 and appearing in Lie algebra commutation relations. The observed Planck constant $\hbar$, whose square defines the scale of M^4 metric corresponds to the ratio of these Planck constants. The key observation is that fractional filling factor results if $\hbar$ is scaled up by a rational number.

In this chapter I try to formulate more precisely this idea. The outcome is a rather detailed view about anyons on one hand, and about the Kähler structure of the generalized imbedding space on the other hand.

1. Fundamental role is played by the assumption that the Kähler gauge potential of CP_2 contains a gauge part with no physical implications in the context of gauge theories but contributing to physics in TGD framework since $U(1)$ gauge transformations are representations of symplectic transformations of CP_2. Also in the case of CD it makes also sense to speak about Kähler gauge potential. The gauge part codes for Planck constants of CD and CP_2 and leads to the identification of anyons as states associated with partonic 2-surfaces surrounding the tip of CD and fractionization of quantum numbers. Explicit formulas relating fractionized charges to the coefficients characterizing the gauge parts of Kähler gauge potentials of CD and CP_2 are proposed based on some empirical input.

2. One important implication is that Poincare and Lorentz invariance are broken inside given CD although they remain exact symmetries at the level of the geometry of world of classical worlds (WCW). The interpretation is as a breaking of symmetries forced by the selection of quantization axis.

3. Anyons would basically correspond to matter at 2-dimensional "partonic" surfaces of macroscopic size surrounding the tip of the light-cone boundary of CD and could be regarded as gigantic elementary particle states with very large quantum numbers and by charge fractionization confined around the tip of CD. Charge fractionization and anyons would be basic characteristic of dark matter (dark only in relative sense). Hence it is not surprising that anyons would have applications going far beyond condensed matter physics. Anyonic dark matter concentrated at 2-dimensional surfaces would play key key role in the the physics of stars and black holes, and also in the formation of planetary system via the condensation of the ordinary matter around dark matter. This assumption was the basic starting point leading to the discovery of the hierarchy of Planck constants [A9]. In living matter membrane like structures would represent a key example of anyonic systems as the model of DNA as topological quantum computer indeed assumes [L5].

4. One of the basic questions has been whether TGD forces the hierarchy of Planck constants realized in terms of generalized imbedding space or not. The condition that the choice of quantization axes has a geometric correlate at the imbedding space level motivated by quantum classical correspondence of course forces the hierarchy: this has been clear from the beginning. It is now clear that also the first principle description of anyons requires the hierarchy in TGD Universe. The hierarchy reveals also new light to the huge vacuum degeneracy of TGD and reduces it dramatically at pages for which CD corresponds to a non-trivial covering or factor space, which suggests that mathematical existence of the theory

necessitates the hierarchy of Planck constants. Also the proposed manifestation of Equivalence Principle at the level of symplectic fusion algebras as a duality between descriptions relying on the symplectic structures of CD and CP_2[C4] forces the hierarchy of Planck constants.

The first sections of the chapter contain summary about theories of quantum Hall effect appearing already in [E9]. Second section is a slightly modified version of the description of the generalized imbedding space, which has appeared already in [A9, E9, L5] and containing brief description of how to understand QHE in this framework. The third section represents the basic new results about the Kähler structure of generalized imbedding space and represents the resulting model of QHE.

2 About theories of quantum Hall effect

The most elegant models of quantum Hall effect are in terms of anyons regarded as singularities due to the symmetry breaking of gauge group G down to a finite sub-group H, which can be also non-Abelian. Concerning the description of the dynamics of topological degrees of freedom topological quantum field theories based on Chern-Simons action are the most promising approach.

2.1 Quantum Hall effect as a spontaneous symmetry breaking down to a discrete subgroup of the gauge group

The system exhibiting quantum Hall effect is effectively 2-dimensional. Fractional statistics suggests that topological defects, anyons, allowing a description in terms of the representations of the homotopy group of $((R^2)^n - D)/S_n$. The gauge theory description would be in terms of spontaneous symmetry breaking of the gauge group G to a finite subgroup H by a Higgs mechanism [39, 30]. This would make all gauge degrees of freedom massive and leave only topological degrees of freedom. What is unexpected that also non-Abelian topological degrees of freedom are in principle possible. Quantum Hall effect is Abelian or non-Abelian depending on whether the group H has this property.

In the symmetry breaking $G \rightarrow H$ the non-Abelian gauge fluxes defined as non-integrable phase factors $Pexp(i\oint A_\mu dx^\mu)$ around large circles (surrounding singularities (so that field approaches a pure gauge configuration) are elements of the first homotopy group of G/H, which is H in the case that H is discrete group and G is simple. An idealized manner to model the situation [30] is to assume that the connection is pure gauge and defined by an H-valued function which is many-valued such that the values for different branches are related by a gauge transformation in H. In the general case a gauge transformation of a non-trivial gauge field by a multi-valued element of the gauge group would give rise to a similar situation.

One can characterize a given topological singularity magnetically by an element in conjugacy class C of H representing the transformation of H induced by a 2π rotation around singularity. The elements of C define states in given magnetic representation. Electrically the particles are characterized by an irreducible representations of the subgroup of $H_C \subset H$ which commutes with an arbitrarily chosen element of the conjugacy class C.

The action of $h(B)$ resulting on particle A when it makes a closed turn around B reduces in magnetic degrees of freedom to translation in conjugacy class combined with the action of element of H_C in electric degrees of freedom. Closed paths correspond to elements of the braid group $B_n(X^2)$ identifiable as the mapping class group of the punctured 2-surface X^2 and this means that symmetry breaking $G \rightarrow H$ defines a representation of the braid group. The construction of these representations is discussed in [30] and leads naturally via the group algebra of H to the so called quantum double $D(H)$ of H, which is a quasi-triangular Hopf algebra allowing non-trivial representations of braid group.

Anyons could be singularities of gauge fields, perhaps even non-Abelian gauge fields, and the latter ones could be modelled by these representations. In particular, braid operations could be represented using anyons.

2.2 Witten-Chern-Simons action and topological quantum field theories

The Wess-Zumino-Witten action used to model 2-dimensional critical systems consists of a 2-dimensional conformally invariant term for the chiral field having values in group G combined with 2+1-dimensional term defined as the integral of Chern-Simons 3-form over a 3-space containing 2-D space as its boundary. This term is purely topological and identifiable as winding number for the map from 3-dimensional space to G. The coefficient of this term is integer k in suitable normalization. k gives the value of central extension of the Kac-Moody algebra defined by the theory.

One can couple the chiral field $g(x)$ to gauge potential defined for some subgroup of G_1 of G. If the G_1 coincides with G, the chiral field can be gauged away by a suitable gauge transformation and the theory becomes purely topological Witten-Chern-Simons theory. Pure gauge field configuration represented either as flat gauge fields with non-trivial holonomy over homotopically non-trivial paths or as multi-valued gauge group elements however remain and the remaining degrees of freedom correspond to the topological degrees of freedom.

Witten-Chern-Simons theories are labelled by a positive integer k giving the value of central extension of the Kac-Moody algebra defined by the theory. The connection with Wess-Zumino-Witten theory come from the fact that the highest weight states associated with the representations of the Kac-Moody algebra of WZW theory are in one-one correspondence with the representations R_i possible for Wilson loops in the topological quantum field theory.

In the Abelian case case 2+1-dimensional Chern-Simons action density is essentially the inner product $A \wedge dA$ of the vector potential and magnetic field known as helicity density and the theory in question is a free field theory. In the non-Abelian case the action is defined by the 3-form

$$\frac{k}{4\pi}Tr\left(A\wedge(dA+\frac{2}{3}A\wedge A)\right)$$

and contains also interaction term so that the field theory defined by the exponential of the interaction term is non-trivial.

In topological quantum field theory the usual n-point correlation functions defined by the functional integral are replaced by the functional averages for $Diff^3$ invariant quantities defined in terms of non-integrable phase factors defined by ordered exponentials over closed loops. One can consider arbitrary number of loops which can be knotted, linked, and braided. These quantities define both knot and 3-manifold invariants (the functional integral for zero link in particular). The perturbative calculation of the quantum averages leads directly to the Gaussian linking numbers and infinite number of perturbative link and not invariants.

The experience gained from topological quantum field theories defined by Chern-Simons action has led to a very elegant and surprisingly simple category theoretical approach to the topological quantum field theory [27, 25] allowing to assign invariants to knots, links, braids, and tangles and also to 3-manifolds for which braids as morphisms are replaced with cobordisms. The so called modular Hopf algebras, in particular quantum groups $Sl(2)_q$ with q a root of unity, are in key role in this approach. Also the connection between links and 3-manifolds can be understood since closed, oriented, 3-manifolds can be constructed from each other by surgery based on links.

Witten's article [26] "Quantum Field Theory and the Jones Polynomial" is full of ingenious constructions, and for a physicist it is the easiest and certainly highly enjoyable manner to learn about knots and 3-manifolds. For these reasons a little bit more detailed sum up is perhaps in order.

1. Witten discusses first the quantization of Chern-Simons action at the weak coupling limit $k\rightarrow\infty$. First it is shown how the functional integration around flat connections defines a topological invariant for 3-manifolds in the case of a trivial Wilson loop. Next a canonical quantization is performed in the case $X^3=\Sigma^2\times R^1$: in the Coulomb gauge $A_3=0$ the action reduces to a sum of $n=dim(G)$ Abelian Chern-Simons actions with a non-linear constraint expressing the vanishing of the gauge field. The configuration space consists thus of flat non-Abelian connections, which are characterized by their holonomy groups and allows Kähler manifold structure.

2. Perhaps the most elegant quantal element of the approach is the decomposition of the 3-manifold to two pieces glued together along 2-manifold implying the decomposition of the functional integral to a product of functional integrals over the pieces. This together with the basic properties of Hilbert of complex numbers (to which the partition functions defined by the functional integrals over the two pieces belong) allows almost a miracle like deduction of the basic results about the behavior of 3-manifold

and link invariants under a connected sum, and leads to the crucial skein relations allowing to calculate the invariants by decomposing the link step by step to a union of unknotted, unlinked Wilson loops, which can be calculated exactly for $SU(N)$. The decomposition by skein relations gives rise to a partition function like representation of invariants and allows to understand the connection between knot theory and statistical physics [24]. A direct relationship with conformal field theories and Wess-Zumino-Witten model emerges via Wilson loops associated with the highest weight representations for Kac Moody algebras.

3. A similar decomposition procedure applies also to the calculation of 3-manifold invariants using link surgery to transform 3-manifolds to each other, with 3-manifold invariants being defined as Wilson loops associated with the homology generators of these (solid) tori using representations R_i appearing as highest weight representations of the loop algebra of torus. Surgery operations are represented as mapping class group operations acting in the Hilbert space defined by the invariants for representations R_i for the original 3-manifold. The outcome is explicit formulas for the invariants of trivial knots and 3-manifold invariant of S^3 for $G = SU(N)$, in terms of which more complex invariants are expressible.

4. For $SU(N)$ the invariants are expressible as functions of the phase $q = exp(i2\pi/(k+N))$ associated with quantum groups. Note that for $SU(2)$ and $k = 3$, the invariants are expressible in terms of Golden Ratio. The central charge $k = 3$ is in a special position since it gives rise to $k + 1 =$ 4-vertex representing naturally 2-gate physically. Witten-Chern-Simons theories define universal unitary modular functors characterizing quantum computations [19].

2.3 Chern-Simons action for anyons

In the case of quantum Hall effect the Chern-Simons action has been deduced from a model of electrons as a 2-dimensional incompressible fluid [29]. Incompressibility requires that the electron current has a vanishing divergence, which makes it analogous to a magnetic field. The expressibility of the current as a curl of a vector potential b, and a detailed study of the interaction Lagrangian leads to the identification of an Abelian Chern-Simons for b as a low energy effective action. This action is Abelian, whereas the anyonic realization of quantum computation would suggest a non-Abelian Chern-Simons action.

Non-Abelian Chern-Simons action could result in the symmetry breaking of a non-Abelian gauge group G, most naturally electro-weak gauge group, to a non-Abelian discrete subgroup H [39] so that states would be labelled by representations of H and anyons would be characterized magnetically H-valued non-Abelian magnetic fluxes each of them defining its own incompressible hydrodynamical flow. As will be found, TGD predicts a non-Abelian Chern-Simons term associated with electroweak long range classical fields.

2.4 Topological quantum computation using braids and anyons

By the general mathematical results braids are able to code all quantum logic operations [23]. In particular, braids allow to realize any quantum circuit consisting of single particle gates acting on qubits and two particle gates acting on pairs of qubits. The coding of braid requires a classical computation which can be done in polynomial time. The coding requires that each dancer is able to remember its dancing history by coding it into its own state.

The general ideas are following.

1. The ground states of anyonic system characterize the logical qubits, One assumes non-Abelian anyons with Z_4 -valued topological charge so that a system of n anyon pairs created from vacuum allows 2^{n-1}-fold anyon degeneracy [37]. The system is decomposed into blocks containing one anyonic Cooper pair with $Q_T \in \{2, 0\}$ and two anyons with such topological charges that the net topological charge vanishes. One can say that the states $(0, 1-1)$ and $(0, -1, +1))$ represent logical qubit 0 whereas the states $(2, -1, -1)$ and $(2, +1, +1)$ represent logical qubit 1. This would suggest 2^2-fold degeneracy but actually the degeneracy is 2-fold.

   Free physical qubits are not possible and at least four particles are indeed necessarily in order to represent logical qubit. The reason is that the conservation of Z^4 charge would not allow mixing of qubits 1 and 0, in particular the Hadamard 1-gate generating square root of qubit would break the conservation of topological charge. The square root of qubit can be generated only if 2 units of topological charge is transferred between anyon and anyon Cooper pair. Thus qubits can be represented as entangled states of anyon Cooper pair and anyon and the fourth anyon is needed to achieve vanishing total topological charge in the batch.

2. In the initial state of the system the anyonic Cooper pairs have $Q_T = 0$ and the two anyons have opposite topological charges inside each block. The initial state codes no information unlike in ordinary computation but the information is represented by the braid. Of course, also more general configurations are possible. Anyons are assumed to evolve like free particles except during swap operations and their time evolution is described by single particle Hamiltonians.

   Free particle approximation fails when the anyons are too near to each other as during braid operations. The space of logical qubits is realized as k-code defined by the 2^{n-1} ground states, which are stable against local single particle perturbations for $k = 3$ Witten-Chern-Simons action. In the more general case the stability against n-particle perturbations with $n < [k/2]$ is achieved but the gates would become $[k/2]$-particle gates (for $k = 5$ this would give 6-particle vertices).

3. Anyonic system provides a unitary modular functor as the S-matrix associated with the anyon system whose time evolution is fixed by the pre-

existing braid structure. What this means that the S-matrices associated with the braids can be multiplied and thus a unitary representation for the group formed by braids results. The vacuum degeneracy of anyon system makes this representation non-trivial. By the NP complexity of braids it is possible to code any quantum logic operation by a particular braid [21]. There exists a powerful approximation theorem allowing to achieve this coding classically in polynomial time [23]. From the properties of the R-matrices inducing gate operations it is indeed clear that two gates can be realized. The Hadamard 1-gate could be realized as 2-gate in the system formed by anyon Cooper pair and anyon.

4. In [18] the time evolution is regarded as a discrete sequence of modifications of single anyon Hamiltonians induced by swaps [20]. If the modifications define a closed loop in the space of Hamiltonians the resulting unitary operators define a representation of braid group in a dense discrete sub-group of $U(2^n)$. The swap operation is 2-local operation acting like a 2-gate and induces quantum logical operation modifying also single particle Hamiltonians. What is important that this modification maps the space of the ground states to a new one and only if the modifications correspond to a closed loop the final state is in the same code space as the initial state. What time evolution does is to affect the topological charges of anyon Cooper pairs representing qubits inside the 4-anyon batches defined by the braids.

   In quantum field theory the analog but not equivalent of this description would be following. Quite generally, a given particle in the final state has suffered a unitary transformation, which is an ordered product consisting of two kinds of unitary operators. Unitary single particle operators $U_n = Pexp(i\int_{t_n}^{t_{n+1}} H_0 dt)$ are analogs of operators describing single qubit gate and play the role of anyon propagators during no-swap periods. Two-particle unitary operators $U_{swap} = Pexp(i\int H_{swap} dt)$ are analogous to four-particle interactions and describe the effect of braid operations inducing entanglement of states having opposite values of topological charge but conserving the net topological charge of the anyon pair. This entanglement is completely analogous to spin entanglement. In particular, the braid operation mixes different states of the anyon. The unitary time development operator generating entangled state of anyons and defined by the braid structure represents the operation performed by the quantum circuit and the quantum measurement in the final state selects a particular final state.

5. Formally the computation halts with a measurement of the topological charge of the left-most anyon Cooper pair when the outcome is just single bit. If decay occurs with sufficiently high probability it is concluded that the value of the computed bit is 0, otherwise 1.

3 A generalization of the notion of imbedding space

In the following the recent view about structure of imbedding space forced by the quantization of Planck constant is described. This view has developed much before the original version of this chapter was written.

The original idea was that the proposed modification of the imbedding space could explain naturally phenomena like quantum Hall effect involving fractionization of quantum numbers like spin and charge. This does not however seem to be the case. $G_a \times G_b$ implies just the opposite if these quantum numbers are assigned with the symmetries of the imbedding space. For instance, quantization unit for orbital angular momentum becomes n_a where Z_{n_a} is the maximal cyclic subgroup of G_a.

One can however imagine of obtaining fractionization at the level of imbedding space for space-time sheets, which are analogous to multi-sheeted Riemann surfaces (say Riemann surfaces associated with $z^{1/n}$ since the rotation by 2π understood as a homotopy of M^4 lifted to the space-time sheet is a non-closed curve. Continuity requirement indeed allows fractionization of the orbital quantum numbers and color in this kind of situation.

3.1 Both covering spaces and factor spaces are possible

The observation above stimulates the question whether it might be possible in some sense to replace H or its factors by their multiple coverings.

1. This is certainly not possible for M^4, CP_2, or H since their fundamental groups are trivial. On the other hand, the fixing of quantization axes implies a selection of the sub-space $H_4 = M^2 \times S^2 \subset M^4 \times CP_2$, where S^2 is a geodesic sphere of CP_2. $\hat{M}^4 = M^4 \backslash M^2$ and $\hat{CP}_2 = CP_2 \backslash S^2$ have fundamental group Z since the codimension of the excluded sub-manifold is equal to two and homotopically the situation is like that for a punctured plane. The exclusion of these sub-manifolds defined by the choice of quantization axes could naturally give rise to the desired situation.

2. H_4 represents a straight cosmic string. Quantum field theory phase corresponds to Jones inclusions with Jones index $\mathcal{M} : \mathcal{N} < 4$. Stringy phase would by previous arguments correspond to $\mathcal{M} : \mathcal{N} = 4$. Also these Jones inclusions are labelled by finite subgroups of $SO(3)$ and thus by Z_n identified as a maximal Abelian subgroup.

   One can argue that cosmic strings are not allowed in QFT phase. This would encourage the replacement $\hat{M}^4 \times \hat{CP}_2$ implying that surfaces in $M^4 \times S^2$ and $M^2 \times CP_2$ are not allowed. In particular, cosmic strings and CP_2 type extremals with M^4 projection in M^2 and thus light-like geodesic without zitterwebegung essential for massivation are forbidden. This brings in mind instability of Higgs=0 phase.

3. The covering spaces in question would correspond to the Cartesian products $\hat{M^4}_{n_a} \times \hat{CP_2}_{n_b}$ of the covering spaces of $\hat{M^4}$ and $\hat{CP_2}$ by Z_{n_a} and Z_{n_b} with fundamental group is $Z_{n_a} \times Z_{n_b}$. One can also consider extension by replacing M^2 and S^2 with its orbit under G_a (say tedrahedral, octahedral, or icosahedral group). The resulting space will be denoted by $\hat{M^4}\hat{\times}G_a$ *resp.* $\hat{CP_2}\hat{\times}G_b$.

4. One expects the discrete subgroups of $SU(2)$ emerge naturally in this framework if one allows the action of these groups on the singular submanifolds M^2 or S^2. This would replace the singular manifold with a set of its rotated copies in the case that the subgroups have genuinely 3-dimensional action (the subgroups which corresponds to exceptional groups in the ADE correspondence). For instance, in the case of M^2 the quantization axes for angular momentum would be replaced by the set of quantization axes going through the vertices of tedrahedron, octahedron, or icosahedron. This would bring non-commutative homotopy groups into the picture in a natural manner.

5. Also the orbifolds $\hat{M^4}/G_a \times \hat{CP_2}/G_b$ can be allowed as also the spaces $\hat{M^4}/G_a \times (\hat{CP_2}\hat{\times}G_b)$ and $(\hat{M^4}\hat{\times}G_a) \times \hat{CP_2}/G_b$. Hence the previous framework would generalize considerably by the allowance of both coset spaces and covering spaces.

There are several non-trivial questions related to the details of the gluing procedure and phase transition as motion of partonic 2-surface from one sector of the imbedding space to another one.

1. How the gluing of copies of imbedding space at $M^2 \times CP_2$ takes place? It would seem that the covariant metric of M^4 factor proportional to $\hbar^2$ must be discontinuous at the singular manifold since only in this manner the idea about different scaling factor of M^4 metric can make sense. This is consistent with the identical vanishing of Chern-Simons action in $M^2 \times S^2$.

2. One might worry whether the phase transition changing Planck constant means an instantaneous change of the size of partonic 2-surface in M^4 degrees of freedom. This is not the case. Light-likeness in $M^2 \times S^2$ makes sense only for surfaces $X^1 \times D^2 \subset M^2 \times S^2$, where X^1 is light-like geodesic. The requirement that the partonic 2-surface X^2 moving from one sector of H to another one is light-like at $M^2 \times S^2$ irrespective of the value of Planck constant requires that X^2 has single point of M^2 as M^2 projection. Hence no sudden change of the size X^2 occurs.

3. A natural question is whether the phase transition changing the value of Planck constant can occur purely classically or whether it is analogous to quantum tunnelling. Classical non-vacuum extremals of Chern-Simons action have two-dimensional CP_2 projection to homologically non-trivial geodesic sphere S^2_I. The deformation of the entire S^2_I to homologically

trivial geodesic sphere S^2_{II} is not possible so that only combinations of partonic 2-surfaces with vanishing total homology charge (Kähler magnetic charge) can in principle move from sector to another one, and this process involves fusion of these 2-surfaces such that CP_2 projection becomes single homologically trivial 2-surface. A piece of a non-trivial geodesic sphere S^2_I of CP_2 can be deformed to that of S^2_{II} using 2-dimensional homotopy flattening the piece of S^2 to curve. If this homotopy cannot be chosen to be light-like, the phase transitions changing Planck constant take place only via quantum tunnelling. Obviously the notions of light-like homotopies (cobordisms) and classical light-like homotopies (cobordisms) are very relevant for the understanding of phase transitions changing Planck constant.

3.2 Do factor spaces and coverings correspond to the two kinds of Jones inclusions?

What could be the interpretation of these two kinds of spaces?

1. Jones inclusions appear in two varieties corresponding to $\mathcal{M} : \mathcal{N} < 4$ and $\mathcal{M} : \mathcal{N} = 4$ and one can assign a hierarchy of subgroups of $SU(2)$ with both of them. In particular, their maximal Abelian subgroups Z_n label these inclusions. The interpretation of Z_n as invariance group is natural for $\mathcal{M} : \mathcal{N} < 4$ and it naturally corresponds to the coset spaces. For $\mathcal{M} : \mathcal{N} = 4$ the interpretation of Z_n has remained open. Obviously the interpretation of Z_n as the homology group defining covering would be natural.

2. $\mathcal{M} : \mathcal{N} = 4$ should correspond to the allowance of cosmic strings and other analogous objects. Does the introduction of the covering spaces bring in cosmic strings in some controlled manner? Formally the subgroup of $SU(2)$ defining the inclusion is $SU(2)$ would mean that states are $SU(2)$ singlets which is something non-physical. For covering spaces one would however obtain the degrees of freedom associated with the discrete fiber and the degrees of freedom in question would not disappear completely and would be characterized by the discrete subgroup of $SU(2)$.

   For anyons the non-trivial homotopy of plane brings in non-trivial connection with a flat curvature and the non-trivial dynamics of topological QFTs. Also now one might expect similar non-trivial contribution to appear in the spinor connection of $\hat{M}^2\hat{\times}G_a$ and $\hat{CP}_2\hat{\times}G_b$. In conformal field theory models non-trivial monodromy would correspond to the presence of punctures in plane.

3. For factor spaces the unit for quantum numbers like orbital angular momentum is multiplied by n_a *resp.* n_b and for coverings it is divided by this number. These two kind of spaces are in a well defined sense obtained by multiplying and dividing the factors of $\hat{H}$ by G_a *resp.* G_b and

multiplication and division are expected to relate to Jones inclusions with $\mathcal{M} : \mathcal{N} < 4$ and $\mathcal{M} : \mathcal{N} = 4$, which both are labelled by a subset of discrete subgroups of SU(2).

4. The discrete subgroups of $SU(2)$ with fixed quantization axes possess a well defined multiplication with product defined as the group generated by forming all possible products of group elements as elements of $SU(2)$. This product is commutative and all elements are idempotent and thus analogous to projectors. Trivial group G_1, two-element group G_2 consisting of reflection and identity, the cyclic groups Z_p, p prime, and tedrahedral, octahedral, and icosahedral groups are the generators of this algebra.

   By commutativity one can regard this algebra as an 11-dimensional module having natural numbers as coefficients ("rig"). The trivial group G_1, two-element group G_2¡ generated by reflection, and tedrahedral, octahedral, and icosahedral groups define 5 generating elements for this algebra. The products of groups other than trivial group define 10 units for this algebra so that there are 11 units altogether. The groups Z_p generate a structure analogous to natural numbers acting as analog of coefficients of this structure. Clearly, one has effectively 11-dimensional commutative algebra in 1-1 correspondence with the 11-dimensional "half-lattice" N^{11} (N denotes natural numbers). Leaving away reflections, one obtains N^7. The projector representation suggests a connection with Jones inclusions. An interesting question concerns the possible Jones inclusions assignable to the subgroups containing infinitely manner elements. Reader has of course already asked whether dimensions 11, 7 and their difference 4 might relate somehow to the mathematical structures of M-theory with 7 compactified dimensions. One could introduce generalized configuration space spinor fields in the configuration space labelled by sectors of H with given quantization axes. By introducing Fourier transform in N^{11} one would formally obtain an infinite-component field in 11-D space.

The question how do the Planck constants associated with factors and coverings relate is far from trivial and I have considered several options.

1. If one assumes that $\hbar^2(X)$, $X = M^4$, CP_2 corresponds to the scaling of the covariant metric tensor g_{ij} and performs an over-all scaling of metric allowed by Weyl invariance of Kähler action by dividing metric with $\hbar^2(CP_2)$, one obtains $r^2 \equiv \hbar^2/\hbar_0^2\hbar^2(M^4)/\hbar^2(CP_2)$. This puts M^4 and CP_2 in a very symmetric role and allows much more flexibility in the identification of symmetries associated with large Planck constant phases.

2. Algebraist would argue that Planck constant must define a homomorphism respecting multiplication and division (when possible) by G_i. This requires $r(X) = \hbar(X)\hbar_0 = n$ for covering and $r(X) = 1/n$ for factor space or vice versa. This gives two options.

3. Option I: $r(X) = n$ for covering and $r(X) = 1/n$ for factor space gives $r \equiv \hbar/\hbar_0 = r(M^4)/r(CP_2)$. This gives $r = n_a/n_b$ for $\hat{H}/G_a \times G_b$ option

and $r = n_b/n_a$ for $\hat{H}\hat{times}(G_a \times G_b)$ option with obvious formulas for hybrid cases.

4. Option II: $r(X) = 1/n$ for covering and $r(X) = n$ for factor space gives $r = r(CP_2)/r(M^4)$. This gives $r = n_b/n_a$ for $\hat{H}/G_a \times G_b$ option and $r = n_a/n_b$ for $\hat{H}\hat{times}(G_a \times G_b)$ option with obvious formulas for the hybrid cases.

5. At quantum level the fractionization would come from the modification of fermionic anti-commutation (bosonic commutation) relations involving $\hbar$ at the right hand side so that particle number becomes a multiple of $1/m$ or m. Partonic 2-surface (wormhole throat) is highly analogous to black hole horizon and this led already years ago the notion of elementary particle horizon and generalization of the area law for black-holes [E5]. The $1/\hbar$-proportionality of the black hole entropy measuring the number of states associated with black hole motivates the hypothesis that the number of states associated with single sheet of the covering proportional to $1/\hbar$ so that the total number states should remain invariant in the transition changing Planck constant. Since the number of states is obviously proportional to the number m of sheets in the covering, this is achieved for $\hbar(X) \propto 1/m$ giving $r(X) \rightarrow r(X)/n$ for factor space and $r(X) \rightarrow nr(X)$ for the covering space. Option II would be selected.

6. The second manner to distinguish between these two options is to apply the theory to concrete physical situations. Since G_a and G_b act as symmetries in M^4 and CP_2 degrees of freedom, one might of being able to distinguish between the two options if it is possible to distinguish between the action of G as symmetry of quantum states associated with covering and factor space. Also the quantization of the orbital spin quantum number at single particle level as multiples of n can be distinguished from that in multiples of $1/n$.

3.3 A simple model of fractional quantum Hall effect

The generalization of the imbedding space suggests that it could possible to understand fractional quantum Hall effect [31] at the level of basic quantum TGD. This section represents the first rough model of QHE constructed for a couple of years ago is discussed. Needless to emphasize, the model represents only the basic idea and involves ad hoc assumption about charge fractionization.

Recall that the formula for the quantized Hall conductance is given by

$$\begin{aligned} \sigma &= \nu \times \frac{e^2}{h} \ , \\ \nu &= \frac{n}{m} \ . \end{aligned} \tag{1}$$

Series of fractions in $\nu = 1/3, 2/5, 3/7, 4/9, 5/11, 6/13, 7/15..., 2/3, 3/5, 4/7, 5/9, 6/11, 7/13...,$ $5/3, 8/5, 11/7, 14/9...4/3, 7/5, 10/7, 13/9..., \ 1/5, 2/9, 3/13..., \ 2/7, 3/11..., \ 1/7....$

with odd denominator have been observed as are also $\nu = 1/2$ and $\nu = 5/2$ states with even denominator [31].

The model of Laughlin [29] cannot explain all aspects of FQHE. The best existing model proposed originally by Jain is based on composite fermions resulting as bound states of electron and even number of magnetic flux quanta [32]. Electrons remain integer charged but due to the effective magnetic field electrons appear to have fractional charges. Composite fermion picture predicts all the observed fractions and also their relative intensities and the order in which they appear as the quality of sample improves.

The generalization of the notion of imbedding space suggests the possibility to interpret these states in terms of fractionized charge, spin, and electron number. There are $2 \times 2 = 4$ combinations of covering and factors spaces of CP_2 and three of them can lead to the increase of Planck constant. Besides this one can consider two options for the formula of Planck constant so that which the very meager theoretical background one can make only guesses. In the following a model based on option II for which the number of states is conserved in the phase transition changing $\hbar$.

1. The easiest manner to understand the observed fractions is by assuming that both M^4 and CP_2 correspond to covering spaces so that both spin and electric charge and fermion number are fractionized. This means that e in electronic charge density is replaced with fractional charge. Quantized magnetic flux is proportional to e and the question is whether also here fractional charge appears. Assume that this does not occur.

2. With this assumption the expression for the Planck constant becomes for Option II as $r = \hbar/\hbar_0 = n_a/n_b$ and charge and spin units are equal to $1/n_b$ and $1/n_a$ respectively. This gives $\nu = nn_a/n_b$. The values $m = 2, 3, 5, 7, ..$ are observed. Planck constant can have arbitrarily large values. There are general arguments stating that also spin is fractionized in FQHE.

3. Both $\nu = 1/2$ and $\nu = 5/2$ state has been observed [31, 33]. The fractionized charge is $e/4$ in the latter case [36, 35]. Since $n_i > 3$ holds true if coverings and factor spaces are correlates for Jones inclusions, this requires $n_a = 4$ and $n_b = 8$ for $\nu = 1/2$ and $n_b = 4$ and $n_a = 10$ for $\nu = 5/2$. Correct fractionization of charge is predicted. For $n_b = 2$ also Z_2 would appear as the fundamental group of the covering space. Filling fraction 1/2 corresponds in the composite fermion model and also experimentally to the limit of zero magnetic field [32]. $n_b = 2$ is inconsistent with the observed fractionization of electric charge for $\nu = 5/2$ and with the vision inspired by Jones inclusions.

4. A possible problematic aspect of the TGD based model is the experimental absence of even values of n_b except $n_b = 2$ (Laughlin's model predicts only odd values of n). A possible explanation is that by some symmetry condition possibly related to fermionic statistics (as in Laughlin model) n_a/n_b must reduce to a rational with an odd denominator for $n_b > 2$. In

other words, one has $n_a \propto 2^r$, where 2^r the largest power of 2 divisor of n_b.

5. Large values of n_a emerge as B increases. This can be understood from flux quantization. One has $e\int BdS = n\hbar(M^4) = nn_a\hbar_0$. By using actual fractional charge $e_F = e/n_b$ in the flux factor would give $e_F\int BdS = n(n_a/n_b)\hbar_0 = n\hbar$. The interpretation is that each of the n_a sheets contributes one unit to the flux for e. Note that the value of magnetic field in given sheet is not affected so that the build-up of multiple covering seems to keep magnetic field strength below critical value.

6. The understanding of the thermal stability is not trivial. The original FQHE was observed in 80 mK temperature corresponding roughly to a thermal energy of $T \sim 10^{-5}$ eV. For graphene the effect is observed at room temperature. Cyclotron energy for electron is (from $f_e = 6\times 10^5$ Hz at $B = .2$ Gauss) of order thermal energy at room temperature in a magnetic field varying in the range 1-10 Tesla. This raises the question why the original FQHE requires so low temperature. The magnetic energy of a flux tube of length L is by flux quantization roughly $e^2B^2S \sim E_c(e)m_eL$ ($\hbar_0 = c = 1$) and exceeds cyclotron roughly by a factor L/L_e, L_e electron Compton length so that thermal stability of magnetic flux quanta is not the explanation. A possible explanation is that since FQHE involves several values of Planck constant, it is quantum critical phenomenon and is characterized by a critical temperature. The differences of the energies associated with the phase with ordinary Planck constant and phases with different Planck constant would characterize the transition temperature.

As already noticed, it is possible to imagine several other options and the assumption about charge fractionization -although consistent with fractionization for $\nu = 5/2$, is rather adhoc. Therefore the model can be taken as a warm-up exercise only.

4 Quantum Hall effect, charge fractionization, and hierarchy of Planck constants

In this section the most recent view about the relationship between dark matter hierarchy and quantum Hall effect is discussed. This discussion leads to a more realistic view about FQHE allowing to formulate precisely the conditions under which anyons emerge, describes the fractionization of electric and magnetic charges in terms of the delicacies of the Kähler gauge potential of generalized imbedding space, and relates the TGD based model to the original model of Laughlin. The discussion allows also to sharpen the vision about the formulation of quantum TGD itself.

4.1 Quantum Hall effect

Recall first the basic facts. Quantum Hall effect (QHE) [29, 30, 31] is an essentially 2-dimensional phenomenon and occurs at the end of current carrying region for the current flowing transversally along the end of the wire in external magnetic field along the wire. For quantum Hall effect transversal Hall conductance characterizing the 2-dimensional current flow is dimensionless and quantized and given by

$$\sigma_{xy} = 2\nu\alpha_{em} \ ,$$

ν is so called filling factor telling the number of filled Landau levels in the magnetic field. In the case of integer quantum Hall effect (IQHE) ν is integer valued. For fractional quantum Hall effect (FQHE) ν is rational number. Laughlin introduced his many-electron wave wave function predicting fractional quantum Hall effect for filling fractions $\nu = 1/m$ [29]. The further attempts to understand FQHE led to the notion of anyon by Wilzeck [30]. Anyon has been compared to a vortex like excitation of a dense 2-D electron plasma formed by the current carriers. ν is inversely proportional to the magnetic flux and the fractional filling factor can be also understood in terms of fractional magnetic flux.

The starting point of the quantum field theoretical models is the effective 2-dimensionality of the system implying that the projective representations for the permutation group of n objects are representations of braid group allowing fractional statistics. This is due to the non-trivial first homotopy group of 2-dimensional manifold containing punctures. Quantum field theoretical models allow to assign to the anyon like states also magnetic charge, fractional spin, and fractional electric charge.

Topological quantum computation [17, 18, E9, L5] is one of the most fascinating applications of FQHE. It relies on the notion of braids with strands representing the orbits of of anyons. The unitary time evolution operator coding for topological computation is a representation of the element of the element of braid group represented by the time evolution of the braid. It is essential that the group involved is non-Abelian so that the system remembers the order of elementary braiding operations (exchange of neighboring strands). There is experimental evidence that $\nu = 5/2$ anyons possessing fractional charge $Q = e/4$ are non-Abelian [36, 35].

During last year I have been developing a model for DNA as topological quantum computer [L5]. Therefore it is of considerable interest to find whether TGD could provide a first principle description of anyons and related phenomena. The introduction of a hierarchy of Planck constants realized in terms of generalized imbedding space with a book like structure is an excellent candidate in this respect [A9]. As a rule the encounters between real world and quantum TGD have led to a more precise quantitative articulation of basic notions of quantum TGD and the same might happen also now.

4.2 TGD description of QHE

The proportionality $\sigma_{xy} \propto \alpha_{em} \propto 1/\hbar$ suggests an explanation of FQHE [30, 29, 31] in terms of the hierarchy of Planck constants. Perhaps filling factors and magnetic fluxes are actually integer valued but the value of Planck constant defining the unit of magnetic flux is changed from its standard value - to its rational multiple in the most general case. The killer test for the hypothesis is to find whether higher order perturbative QED corrections in powers of α_{em} are reduced from those predicted by QED in QHE phase. The proposed general principle governing the transition to large $\hbar$ phase is states that Nature loves lazy theoreticians: if perturbation theory fails to converge, a phase transition increasing Planck constant occurs and guarantees the convergence. Geometrically the phase transition corresponds to the leakage of 3-surface from a given 8-D page to another one in the Big Book having singular coverings and factor spaces of $M^4 \times CP_2$ as pages.

Chern-Simons action for Kähler gauge potential (equivalently for induced classical color gauge field proportional to the Kähler form) defines TGD as almost topological QFT. This alone strongly suggests the emergence of quantum groups and fractionalization of quantum numbers. The challenge is to figure out the details and see whether this framework is consistent with what is known about QHE. At least the following questions pop up immediately in mind.

1. What the effective 2-dimensionality of the system exhibiting QHE corresponds in TGD framework?

2. What happens in the phase transition leading to the phase exhibiting QHE effect?

3. What are the counterparts anyons? How the fractional electric and magnetic charges emerge at classical and quantum level.

4. The Chern-Simons action associated with the induced Kähler gauge potential is Abelian: is this consistent with the non-Abelian character of braiding matrix?

4.3 Quantum TGD almost topological QFT

The statement that TGD is almost topological QFT means following conjectures.

1. In TGD the fundamental physical object is light-like 3-surface X^3 connecting the light-cone boundaries of $CD \times CP_2 \subset M^4 \times CP_2$ (intersection of future and past directed light-cones) but by conformal invariance in the light-like direction of X^3 physics is locally 2-dimensional in the sense that one can regard this surface as an orbit of 2-D parton as long as one restricts to finite region of X^3. Physics at X^3 remains 3-D in discretized sense (quantum states are of course quantum superpositions of different light-like 3-surfaces).

2. At the fundamental level quantum TGD can be formulated in terms of the fermionic counterpart of Chern-Simons action fort the Kähler gauge potential associated with Kähler form of CP_2. The Dirac determinant associated with the modified Dirac action defines the vacuum functional of the theory. Dirac determinant is defined as a finite product of the values of generalizes eigenvalues (functions) of the modified Dirac operator at points defined by the strands of so called number theoretic braids which by number theoretic arguments are unique [E1, E2].

3. Vacuum functional equals to the exponent of Kähler action for a preferred extremal $X^4(X^3)$ of Kähler action, which plays the role of Bohr orbit and allows to realize 4-D general coordinate invariance. The boundary conditions of 4-D dynamics fixing $X^4(X^3)$ are fixed by the requirement that the tangent space of X^4 contains a preferred Minkowski plane $M^2 \subset M^4$ at each point. This plane can be interpreted as the plane of non-physical polarizations.

4. "Number theoretic compactification" states that space-time surfaces can be regarded as 4-surfaces of either hyper-octonionic M^8 or $M^4 \times CP_2$ (hyper-octonions corresponds to a sub-space of complexified octonions with Minkowskian signature of metric). The surfaces of M^8 are hyper-quaternionic in the sense that each tangent plane is hyper-quaternionic and contains (this is essential for number theoretic compactification) the preferred hyper-complex plane M^2 defined by hyper-octonionic real unit and preferred imaginary unit. The preferred extrema of Kähler action should correspond hyper-quaternionic 4-surfaces of M^8 having preferred M^2 as a tangent space at each point.

These 'must-be-trues' are of course highly non-trivial un-proven conjectures. If one gives up conjecture about the reduction of entire 4-D dynamics to that for almost topological fermions at 3-D light-like surfaces, one must assume separately that vacuum functional is exponent of Kähler function for a preferred extremal.

4.4 Constraints to the Kähler structure of generalized imbedding space from charge fractionization

In the following the notion of generalized imbedding space is discussed. The new element is more precise formulation of the Kähler structure by allowing Kähler gauge potential to have what looks formally as gauge parts in both M^4 and CP_2 and of no physical significance on gauge theory context. In TGD framework the gauge parts have deep physical significance since symplectic transformations act as symmetries of Kähler and Chern-Simons-Kähler action only in the case of vacuum extremals.

4.4.1 Hierarchy of Planck constants and book like structure of imbedding space

TGD leads to a description for the hierarchy of Planck constants in terms of the generalization of the Cartesian factors of the imbedding space $H = M^4 \times CP_2$ to book like structures. To be more precise, the generalization takes place for any region $CD \times CP_2 \subset H$, where CD corresponds to a causal diamond defined as an intersection of future and past directed light-cones of M^4. CDs play key role in the formulation of quantum TGD in zero energy ontology in which the light-like boundaries of CD connected by light-like 3-surfaces can be said to be carriers of positive and negative energy parts of zero energy states. They are also crucial for TGD inspired theory of consciousness, in particular for understanding the relationship between experienced and geometric time [16].

Both CD and CP_2 are replaced with a book like structure consisting in the most general case of singular coverings and factor spaces associated with them. A simple geometric argument identifying the square of Planck constant as scaling factor of the covariant metric tensor of M^4 (or actually CD) leads to the identification of Planck constant as the ratio $\hbar/\hbar_0 = q(M^4)/q(CP_2)$, where $q(X) = N$ holds true for the covering of X and $q(X) = 1/N$ holds true for the factor space. N is the order of the maximal cyclic subgroup of the covering/divisor group $G \subset SO(3)$. The order of G can be thus larger than N. As a consequence, the spectrum of Planck constants is in principle rational-valued. $\hbar_0$ is unique since it corresponds to the unit of rational numbers. The field structure has far reaching implications for the understanding of phase transitions changing the value of Planck constant.

The hierarchy of Planck constants relates closely to quantum measurement theory. The selection of quantization axis has a direct correlate at the level of imbedding space geometry. This means breaking of isometries of H for a given CD with preferred choice time axis (rest frame) and quantization axis of spin. For CP_2 the choice of the quantization axes of color hyper charge and isospin imply symmetry breaking $SU(3) \rightarrow U(2) \rightarrow U(1) \times U(1)$. The "world of classical worlds" (WCW) is union over all Poincare and color translates of given $CD \times CP_2$ so that these symmetries are not lost at the level of WCW although the loss can happen at the level of quantum states.

4.4.2 Non-vanishing of Poincare quantum numbers requires CP_2 Kähler gauge potential to have M^4 part

Since Kähler action gives rise to conserved Poincare quantum numbers as Noether charges, the natural expectation is that Poincare quantum numbers make sense as Noether charges for Chern-Simons action. The problem is that Poincare quantum numbers vanish for standard Kähler gauge potential of CP_2 since it has no M^4 part.

The way out of the difficulty relies on the delicacies of CP_2 Kähler structure.

1. One can give up the strict Cartesian product property and assume that CP_2 Kähler gauge potential has M^4 part which is pure gauge and with-

out physical meaning in gauge theory context. In TGD framework the situation is different. The reason is that $U(1)$ gauge transformations are induced by the symplectic transformations of CP_2 and correspond to genuine dynamical symmetries acting as isometries of WCW. They act as symmetries of Kähler action only in the case of vacuum extremals and relate closely to the spin glass degeneracy of Kähler action with the counterpart of spin glass energy landscape defined by small deformations of vacuum extremals of Kähler action. This vacuum degeneracy has been one of the most fruitful challenges of TGD.

2. Requiring Lorentz invariance one can write the non-vanishing pure gauge M^4 component of Kähler gauge potential as

$$A_a = constant \ . \tag{2}$$

Here a denotes the light-cone proper time. It is of course possible that also other components are present as it indeed turns out. Using standard formula for Noether current one finds that four-momentum is non-vanishing because of the term $A_a\partial_\alpha a$ in Chern-Simons-Kähler action. From $\partial_\alpha a = m^k m_{kl}\partial_\alpha m^l/a$ momentum current T^{k0} at given point of X^3 is proportional to the average 4-velocity with respect to the tip of light-cone: $T^{k0} \propto m^k/a$. Therefore the motion in the average sense is analogous to cosmic expansion. This is natural since the structure of CD corresponding to particular quantization axes breaks Poincare symmetry.

3. $A_a = constant$ guarantees the conservation of mass squared in the case of CP_2 type extremals at least and implies that mass squared is non-vanishing. Four-momentum is also proportional to the Kähler magnetic flux over the partonic 2-surface X^2 and X^2 must be homologically non-trivial for the net value of four-momentum to be non-vanishing. X^2 could correspond to the end of cosmic string in 4-D picture. Homological non-triviality does not seem to be necessary in the case of super-symmetric counterpart of Dirac action since Kähler flux is multiplied by the fermionic bilinear so that the outcome is more general than Kähler magnetic flux.

4.4.3 The M^4 part of CP_2 Kähler gauge potential for the generalized imbedding space

The non-triviality of A_a transforms topological QFT to an almost topological one, but says nothing about the covering- and factor space sectors of generalized imbedding space- the pages of the book like structure defined by the generalized imbedding space. The interpretation in terms of quantum measurement theory suggests that Lorentz symmetry and color symmetry are broken to Cartan subgroups defining quantization axes. If anyons correspond to large $\hbar$ phase, the Kähler gauge potential of CP_2 should contain in these sectors additional gauge parts in both M^4 and CP_2 responsible for charge fractionization, magnetic monopoles, and other anyonic effects.

The basic prerequisite for anyonic effects is that fundamental group is non-trivial and for M^4 the emergence of M^2 as the intersection of sheets of the singular covering implies this for the complement of M^2. In the case of CP_2 the homologically trivial geodesic S^2 is common to the coverings and factors spaces and implies the non-triviality of the fundamental group.

Let $(u = m^0 + r_M), v = m^0 - r_M, \theta, \phi)$ define light-like spherical coordinates for $M^4_\pm$. Here m^k are linear M^4 time coordinates and r_M is radial M^4 coordinate. Denote the light-cone proper time by $a = \sqrt{uv}$. The origin of coordinates lies at the either tip of CD. Coordinates are not global so that the patches assignable to positive and negative energy parts of the zero energy state must be used.

The fixing of the rest system, that is the direction of time axis, reduces Lorentz invariance to $SO(3)$. This allows A to have an additional part

$$A_u = \frac{k_1}{u^2} . \tag{3}$$

The functional form of A_u will be deduced in the sequel from the conservation of anyonic charges. The fixing of the direction of the spin quantization axis reduces the symmetry to $SO(2)$ and allows introduction of a further gauge component

$$A_\phi = \frac{k_2}{u^2} . \tag{4}$$

Clearly one has a hierarchical breaking of symmetry: Poincare group → Lorentz group→ rotation group $SO(3) \rightarrow SO(2)$. Globally the symmetry is not broken since WCW is a union over all possible choices of quantization for each CDs with all possible positions of lower tip are allowed. p-Adic length scale hypothesis results if the temporal distance between upper and lower tips is quantized in multiples 2^n. The hierarchy of Planck constants however implies that distance are quantized as rational multiples of basic distance scale.

4.4.4 How fractional electric and magnetic charges emerge from M^4 gauge part of CP_2 Kähler gauge potential?

The Maxwell field defined by the induced CP_2 Kähler form plays fundamental role in the construction of quantum TGD. Kähler gauge potential of CP_2 contributes directly to the classical electromagnetic gauge potential. Its coupling to $M^4 \times CP_2$ spinors is different for quarks and leptons representing different conserved chiralities of H spinors and it explains different electromagnetic charges of quarks and leptons as well as different color trialities. Also classical color gauge field is proportional to Kähler form. Therefore one might hope that the gauge parts of Kähler gauge potential might contain a lot of interesting physics.

The following series of arguments try to demonstrate following three results.

1. The anomalous contribution to the Kähler gauge potential induces anomalous electric and magnetic Kähler charges and therefore also em, Z^0, and color gauge charges.

2. Anyons can be characterized as 2-surfaces surrounding the tip of CD.

3. In sectors corresponding to the non-standard value of $\hbar$ the vacuum degeneracy of Kähler and Chern-Simons actions is dramatically reduced.

Note that in this section the consideration is restricted to the gauge parts of CP_2 Kähler gauge potential in $CD \subset M^4$. Also the gauge parts in CP_2 are possible and the Kähler potential assignable to the contact structure of CD must be considered separately.

1. The gauge part of Kähler gauge potential vanishes outside CD so that it is discontinuous at light-like boundary in the direction of the light like vector defined as the gradient of $v = t - r$. This means that for partonic 2-surfaces surrounding the tip of light-cone both Kähler electric and magnetic fluxes are non-vanishing and determined by $K_i(u)$, $i = 1, 2$. By requiring that the anomalous Kähler charge is time independent, one obtains $K_1(u) = k_1/u^2$. This means that the Kähler electric gauge field has a delta function like singularity at the light-like boundares of CD which becomes carrier of Kähler charge from the view point of complement of CD. This suggests that if one has N elementary particles at partonic 2-surface X^2 surrounding the tip of CD (wormhole throats of elementary particles are condensed to X^2), the charges of particles are effectively fractionized:

$$q \rightarrow q + \frac{Q_A}{N} . \tag{5}$$

2. In the case of $A_\phi = constant$ anomalous magnetic charge results since the flux expressible as line integral $\int A_\phi d\phi$ is non-vanishing because the poles of S^2 act effectively as magnetic charges. The punctures at the poles are the correlate for the selection of the quantization axes of spin. $K_2(u) = k_2/u^2$ follows from the conservation of magnetic charge. In the case of ordinary magnetic monopole spin becomes half-odd integer valued and analogous result holds also now. The minimal coupling to the gauge part of A_ϕ defining the covariant derivative D_ϕ together with covariant constancy condition implies that spin receives a fractional part for $k_2 \neq 0$ and spin fractionization results.

3. One can see the situation also differently. The 2-D partons at the ends of light-like 3-surfaces at light-like boundaries of CD interact like particles with anomalous gauge charges but the interaction is now in light-like direction. The anomalous charges indeed characterize Chern-Simons action. For $k_1 = k_2 = 0$ corresponding to $\hbar/\hbar_0 = 1$ one has Lorentz invariance and only cosmic string like objects seem to remain to the spectrum of the theory (they dominate the very early TGD based cosmology [D6]).

4. Quite generally, anyonic states can be assigned with partonic 2-surfaces surrounding surrounding the tip of CD since the fractional contribution to the gauge charge vanishes otherwise.

5. Kähler gauge potential appears in the expression of the em charge so that a fractionization of electric and magnetic em and Z^0 charges results but there is no fractionization of the weak charge. The components of the classical color gauge field are of form $G^A \propto H^A J$, H^A the Hamiltonian of color isometry and J Kähler form. The assumption that the singular part of G^A is induced from that for J implies anomalous electric and magnetic color gauge charges located at boundaries of CD. These charges should make sense as fluxes since the $SU(3)$ holonomy is Abelian.

6. A_u contributes to the four-momentum density a term proportional to the four-vector $\partial u/\partial m^k$ which in vector notation looks like $(1, \overline{r}_M/r_M)$: thus the direction of 3-momentum rends to be same as for A_a. In the approximation that the M^4 coordinates for partonic 2-surface are constant (excellent approximation at elementary particle level) this contribution to the four-momentum is massless unlike for A_a. If the variation of the projection of A_u in Chern-Simons action is responsible for the four-momentum X^2 must carry non-vanishing homological charge for Chern-Simons action but not for its fermionic counterpart. If the variation of the projection of the singular part J_{uv} is responsible for the momentum the CP_2 projection can be 1-dimensional so that the vacuum degeneracy is reduced and the homological non-triviality in CP_2 is replaced with homological triviliaty in CD with the line connecting the tips of CD removed.

7. For $(k_1, k_2) = (0, 0)$ all space-time surfaces for which CP_2 projection is Lagrange manifold of CP_2 (generally 2-dimensional sub-manifold having vanishing induced Kähler form) are vacuum extremals For $(k_1, k_2) \neq (0, 0)$ and for partonic 2-surfaces surrounding the tip of the light-cone, the situation changes since also partonic 2-surfaces which have 1-D CP_2 projection can carry non-vanishing Kähler, em, and color charges, and even four-momentum. If M^4 projection is 2-D, the anomalous part of Kähler form contributing to the charges is completely in M^4 and the variation of of A_α in Chern-Simons action gives rise to color currents. Four-momentum can be non-vanishing even when CP_2 projection is zero-dimensional since the variation of A_a gives rise to it when X^2 surrounds the tip of CD. Hence the hierarchy of Planck constants removes partially the vacuum degeneracy. This correlation conforms with the general idea that both the vacuum degeneracy and the hierarchy of Planck constants relate closely to quantum criticality. Perhaps the hierarchy of Planck constants accompanied by the anyonic gauge parts of A makes possible to have mathematically well-define theory.

4.4.5 Coverings and factor spaces of CP_2 and anyonic gauge part of Kähler gauge potential in CP_2?

Nothing about possible coverings and factor spaces of CP_2 has been said above. In principle they could contribute to CP_2 Kähler gauge potential an anomalous part and would form a representation for the hierarchy of Planck constants in CP_2 degrees of freedom.

1. If Kähler gauge potential has also anyonic CP_2 part, it should fix the choice of quantization axes for color charges. Thus the anomalous components could be of form $A_{I_3} = k(I_3)$ and $A_Y = k(Y_3)$ where the angle variables vary along flow lines of I_3 and Y. Singularity would emerge both at the origin and at the 2-sphere $r = \infty$ analogous to the North pole of S^2, at which the second angle variable becomes redundant.

2. These terms would give to the anomalous Kähler magnetic charge a contribution completely analogous to that coming from A_ϕ. Also color charges would receive similar contribution.

4.4.6 How the values of the anomalous charges relate to the parameters characterizing the page of the Big Book?

One should be able to relate the anomalous parameters characterizing anomalous gauge potentials to the parameters n_a, n_b characterizing the coverings of CD and CP_2. Consider first various manners to understand charge fractionization.

1. The hypothesis at the end of previous section states that for n_b-fold covering of CP_2 the fractionized electric charge equals to e/n_b. This predicts charge fractionization correctly for $\nu = 5/2 = 10/4$ [**?**]. This simple argument could apply also to other charges. The interpretation would be that when elementary particle becomes anyonic, its charge is shared between n_b sheets of the covering of CP_2. In the case of factor space the singular factor space would appear as n_b copies meaning the presence n_b particles behaving like single particle. Charge fractionization would be only apparent in this picture.

2. This global representation of the fractionization of Kähler charge might be enough. One can however ask whether also a local representation could exist in the sense that the coupling of fermions to the gauge parts of Kähler gauge potential would represent charge fractionization at single particle level in terms of phase factors analogous to plane waves. If charge fractionization is only apparent, the total anomalous Kähler charge assignable to particles should be compensated by the total anomalous Kähler charge associated with A_u. This gives a constraint between k_1 and parameter $k(Y)$.

3. Similar argument for the Kähler magnetic charge gives a constraint between k_2 and $k(Y)$ implying $k_1 = k_2$ consistent with assumption that also the anyonic part of Kähler form is self dual. In the simplest situation $k_1 = k_2 = Nq_K k(Y)$, where N is the number of identical particles at the anyonic space-time sheet. In more general case one would have $k_1 = k_2 \sum_i N_i q_{K,i} k(Y)$. If the anyonic space-time sheet does not contain the tip of CD in its interior, the total anomalous Kähler charge associated with the fermions at it must vanish.

4. Both em and Z^0 fields contain a part proportional to Kähler form so that total anomalous gauge charges defined as fluxes should be equal to those defined as sums of elementary particle contributions.

5. Anomalous color isospin and hypercharge and corresponding magnetic charges would have also representations as color gauge fluxes by using $Q^A \propto H^A J$ restricted to Cartan algebra of color group. The couplings to the anomalous gauge parts of Kähler gauge potential in CP_2 would give rise to anomalous color charges at single particle level, and also now the condition that the total anomalous charges assignable to particles compensates that assignable to the singular part of color gauge potential is natural. Thus quite a number of consistency conditions emerge.

The foregoing discussion relates to the gauge part of Kähler gauge potential assigned to CP_2 degrees of freedom. Analogous discussion applies to the M^4 part.

1. Covariant constancy conditions appear also in Minkowski degrees of freedom and correlate the value of anomalous Poincare charges to anomalous Kähler charge. Anomalous Kähler charge k_1 gives via covariant constancy condition for induced spinors contribution to four-momentum analogous to Coulomb interaction energy with Kähler charge k_1: at point like limit the contribution is light-like. In the similar manner $k_2 = k_1$ gives rise to anomalous orbital spin via the covariant constancy condition $D_\phi \Psi = (\partial_\phi + A_\phi)\Psi = 0$ equating A_ϕ with the fractional contribution to spin. Thus both anomalous four-momentum and spin fractionization effect reflects the total anomalous Kähler charge.

2. The values of $k_1 = k_2$ should correlate directly with the order of the maximal cyclic subgroup Z_{n_a} associated with the covering/factor space of CD. For covering one should should have $k_2 = n/n_a$ since the rotation by $N \times 2\pi$ is identity transformation. For the factor space one should have $k_2 = nn_a$ since the states must remain invariant under rotations by multiples of $2\pi/N$ and spin unit becomes n_a. This picture is consistent with the scaling up of the spin unit with $\hbar/\hbar_0$. Since k_1 must be also an integer multiple of $1/n_b$, k_1 should be inversely proportional to a common factor of n_a and n_b.

That classical color hyper charge and isospin correspond to electro-weak charges is an old idea which I have not been able to kill. It is discussed also in [C4] from the point of view of symplectic fusion algebras.

1. Quark color is not a spin like quantum number but corresponds to CP_2 partial waves in cm degrees of freedom of partonic 2-surface. Hence it should not relate to the classical color charges associated with classical color gauge field or with the modes of induced spinor fields at space-time sheet. These nodes can also carry color hyper charge and isospin in the sense that they are proportional to space-time projections of phase factors representing states with constant Y and I_3 (being completely analogous to angular momentum eigen states on circle).

2. In the construction of symmetric spaces the holonomy group of the spinor connection is identified as a subgroup of the isometry group. Therefore electro-weak gauge group $U(2)_{ew}$ would correspond to $U(2) \subset SU(3)$ defining color quantization axis. If so, the phase factors assignable to the induced spinor fields could indeed represent the electromagnetic and weak charges of the particle and one would have $Y = Y_{ew}$ and $I_3 = I_{3,ew}$. Also electro-weak quantum numbers, which are spin-like, would have geometric representation as phase factors of spinors.

3. This kind of multiple representation emerges also via number theoretical compactication [E2] meaning that space-time surfaces can be regarded either as surfaces in hyper-octonionic space $M^8 = M^4 \times E^4$ or $M^4 \times CP_2$. In M^8 electro-weak quantum numbers are represented as particle waves and color is spin like quantum number.

Again a word of caution is in order since the formula for charge fractionization is supported only by its success in $\nu = 5/2$ case. Also the proposed formulas are only heuristic guesses.

4.4.7 What about Kähler gauge potential for CD?

One can assign also to light-cone boundary- metrically equivalent with S^2, symplectic (or more precisely contact-) structure. This structure can be extended to a pseudo-symplectic structure in the entire CD. The structure is not global and one must introduce two patches corresponding to the two light-cone boundaries of CD.

This symplectic structure plays a key role in the construction of symplectic fusion algebra [C4]. In TGD framework Equivalence Principle is realized in terms generalized coset construction for the super-canonical conformal algebra assignable to the light-cone boundary and super-Kac-Moody algebra assignable to the light-like 3-surfaces. The cautious proposal of [C4] is that at the level of fusion algebra Equivalence Principle means the possibility to use either the symplectic fusion algebra of light-cone boundary for light-cone defined by S^2 Kähler form or the symplectic fusion algebra for light-cone boundry defined by CP_2 Kähler form.

The vacuum degeneracy of Kähler action requiring that CP_2 projection of the partonic 2-surface is non-trivial would at first seem to exclude this option. Anomalous gauge charges however remove this vacuum degeneracy for $k_1 \neq 0$ so that there are no obvious reasons excluding this manifestation of Equivalence Principle.

The Kähler gauge potential of the degenerate Kähler form assignable to the light-like boundary (basically to the $r_M = constant$ sphere S^2) and also to CD and identifiable as the Kähler form of S^2 defining its signed area can indeed contain gauge part with a structure similar that for CP_2 Kähler gauge potential and involving three rational valued constants corresponding to gauge parts A_a, A_u, and A_ϕ. The TGD based realization of the Equivalence Principle suggests that the constants associated with the two Kähler forms are identical or at least proportional to each other. One could perhaps even say that the hierarchy of Planck constants and dark matter are necessary to realize Equivalence Principle in TGD framework.

4.5 In what kind of situations do anyons emerge?

Charge fractionization is a fundamental piece of quantum TGD and should be extremely general phenomenon and the basic characteristic of dark matter known to contribute 95 per cent to the matter of Universe.

1. In TGD framework scaling $\hbar = m\hbar_0$ implies the scaling of the unit of angular momentum for m-fold covering of CD only if the many particle state is Z_m singlet. Z_m singletness for many particle states allows of course non-singletness for single particle states. For factor spaces of CD the scaling $\hbar \rightarrow \hbar/m$ is compensated by the scaling $l \rightarrow ml$ for $L_z = l\hbar$ guaranteing invariance under rotations by multiples of $2\pi/m$. Again one can pose the invariance condition on many-particle states but not to individual particles so that genuine physical effect is in question.

2. There is analogy with Z_3-singletness holding true for many quark states and one cannot completely exclude the possibility that quarks are actually fractionally charged leptons with $m = 3$-covering of CP_2 reducing the value of Planck constant [A8, A9] so that quarks would be anyonic dark matter with smaller Planck constant and the impossibility to observe quarks directly would reduce to the impossibility for them to exist at our space-time sheet. Confinement would in this picture relate to the fractionization requiring that the 2-surface associated with quark must surround the tip of CD. Whether this option really works remains an open question. In any case, TGD anyons are quite generally confined around the tip of CD.

3. Quite generally, one expects that dark matter and its anyonic forms emerge in situations where the density of plasma like state of matter is very high so that N-fold cover of CD reduces the density of matter by $1/N$ factor at given sheet of covering and thus also the repulsive Coulomb energy.

Plasma state resulting in QHE is one examples of this. The interiors of neutron stars and black hole like structures are extreme examples of this, and I have proposed that black holes are dark matter with a gigantic value of gravitational Planck constant implying that black hole entropy -which is proportional to $1/\hbar$ - is of same order of magnitude as the entropy assignable to the spin of elementary particle. The confinement of matter inside black hole could have interpretation in terms of macroscopic anyonic 2-surfaces containing the topologically condensed elementary particles. This conforms with the TGD inspired model for the final state of star [D4] inspiring the conjecture that even ordinary stars could posses onion like structure with thin layers with radii given by p-adic length scale hypothesis. The idea about hierarchy of Planck constants was inspired by the finding that planetary orbits can be regarded as Bohr orbits [40, A9]: the explanation was that visible matter has condensed around dark matter at spherical cells or tubular structures around planetary orbits. This led to the proposal that planetary system has formed through this kind of condensation process around spherical shells of dark matter. The question why dark matter would concentrate at spherical shells was not answered. The answer would be that dark matter is anyonic matter at these 2-surfaces.

4. DNA as topological quantum computer idea assumes that DNA nucleotides are connected by magnetic flux tubes to the lipids of the cell membrane. In this case, p-adically scaled down u and d quarks and their antiquarks are assumed to be associated with the ends of the flux tubes and provide a representation of DNA nucleotides. Quantum Hall states would be associated with partonic 2-surfaces assignable to the lipid layers of the cell and nuclear membrane and also endoplasmic reticulum filling the cell interior and making it macroscopic quantum system and explaining also its stability.

4.6 What happens in QHE?

This picture suggest following description for what would happens in QHE in TGD Universe.

1. Light-like 3-surfaces - locally random light-like orbits of partonic 2-surfaces- are identifiable as very tiny wormhole throats in the case of elementary particles. This is the case for electrons in particular. Partonic surfaces can be also large, even macroscopic, and the size scales up in the scaling of Planck constant. To avoid confusion, it must be emphasized that light-likeness is with respect to the induced metric and does not imply expansion with light velocity in Minkowski space since the contribution to the induced metric implying light-likeness typically comes from CP_2 degrees of freedom. Strong classical gravitational fields are present near the wormhole throat. Second important point is that regions of space-time surface with Euclidian signature of the induced metric are implied:

CP_2 type extremals representing elementary particles and having light-like random curve as CP_2 projection represents basic example of this. Hence rather exotic gravitational physics is predicted to manifest itself in everyday length scales.

2. The simplest identification for what happens in the phase transition to quantum Hall phase is that the end of wire carrying the Hall current corresponds to a partonic 2-surface having a macroscopic size. The electrons in the current correspond to similar 2-surfaces but with size of elementary particle for the ordinary value of Planck constant. As the electrons meet the end of the wire, the tiny wormhole throats of electrons suffer topological condensation to the boundary. One can say that one very large elementary particle having very high electron number is formed.

3. The end of the wire forms part of a spherical surface surrounding the tip of the CD involved so that electrons can become carriers of anomalous electric and magnetic charges.

4. Chern- Simons action for Kähler gauge potential is Abelian. This raises the question whether the representations of the number theoretical braid group are also Abelian. Since there is evidence for non-Abelian anyons, one might argue that this means a failure of the proposed approach. There are however may reasons to expect that braid group representations are non-Abelian. The action is for induced Kähler form rather than primary Maxwell field, $U(1)$ gauge symmetry is transformed to a dynamical symmetry (symplectic transformations of CP_2 representing isometries of WCW and definitely non-Abelian), and the particles of the theory belong to the representations of electro-weak and color gauge groups naturally defining the representations of braid group.

5. The finite subgroups of $SU(2)$ defining covering and factor groups are in the general case non-commutative subgroups of $SU(2)$ since the hierarchies of coverings and factors spaces are assumed to correspond to the two hierarchy of Jones inclusions to which one can assign ADE Lie algebras by McKay correspondence. The ADE Lie algebras define effective gauge symmetries having interpretation in terms of finite measurement resolution described in terms of Jones inclusion so that extremely rich structures are expected.

6. The proposed model allows charge and spin fractionization also for IQHE since $\hbar/\hbar_0 = 1$ holds true for $n_a = n_b$. There is also infinite number of anyonic states predicting a given value of ν ($(n_a, n_b) \rightarrow k(n_a, n_b)$ symmetry).

An interesting challenge is to relate concrete models of QHE to the proposed description. Here only some comments about Laughlin's wave function are made.

1. In the description provided by Lauglin wave function FQHE results from a minimization of Coulomb energy. In TGD framework the tunneling to the page of H with m sheets of covering has the same effect since the density of electrons is reduced by $1/m$ factor.

2. The formula $\nu \propto e^2 N_e/e \int BdS$ with scaling up of magnetic flux by $\hbar/\hbar_0 = m$ implies effective fractional filling factor. The scaling up of magnetic flux results from the presence of m sheets carrying magnetic field with same strength. Since the N_e electrons are shared between m sheets, the filling factor is fractional when one restricts the consideration to single sheet as one indeed does.

3. Laughlin wave function makes sense for $\nu = 1/m$, m odd, and is m:th power of the many electron wave function for IQHE and expressible as the product $\prod_{i<j}(z_i - z_j)^m$, where z represents complex coordinate for the anyonic plane. The relative orbital angular momenta of electrons satisfy $L_z \geq m$ if the value of Planck constant is standard. If Laughlin wave function makes sense also in TGD framework, then m:th power implies that many-electron wave function is singlet with respect to Z_m acting in covering and the value of relative angular momentum indeed satisfies $L_z \geq m\hbar_0$ just as in Laughlin's theory.

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